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On normal vorticity and current density generated behind a magnetogasdynamic shock wave in unsteady flows

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Two interesting linear relations between normal vorticity and current density generated behind a shock wave in three-dimensional unsteady flows of conducting gases have been investigated in order to determine them purely from dynamical considerations. Some interesting physical conclusions have also been made.

1. Introduction

Vorticity generated behind a shock wave has been a subject of interest discussed by several authors. Truesdell (1952) concluded for plane shocks in ordinary gasdynamics that *vorticity generated by a shock of given strength and curvature depends only on the magnitude of the tangential components of velocity and is independent of the form of the equation of state.* Lighthill (1957) derived an expression for the vorticity generated behind a shock in steady flows of gases obeying an arbitrary equation of state. In accordance with Truesdell's statement Hayes (1957) provided a proof based on the momentum equation alone for the derivation of vorticity jump across a gasdynamic discontinuity. For planar shocks in stationary flows of an ideal conducting gas Kanwal (1960*a*) concluded that *vorticity generated behind a shock in conducting gases is no longer derivable purely from dynamical relations as in non-conducting gases.* Ram (1967) extended this result to the general case of unsteady flows of conducting gases by employing an easier method. Ram & Mishra (1966) derived the normal vorticity and current density behind a magnetogasdynamic shock wave purely from dynamical considerations in the case of pseudo-stationary flows. The present paper generalizes this to the case of unsteady flows of conducting gases.

Let the shock configuration in 3-dimensional unsteady flows be represented by continuously differentiable functions $x_i = x_i(y^\alpha, t)$,† where x_i are the rectangular Cartesian co-ordinates of a point P on the shock surface and y^α are the Gaussian co-ordinates of P. Let G be the speed of the shock and N_i be the components of the unit normal to the shock surface directed downstream. Let $[f]$ denote the jump in the quantity enclosed as it crosses the shock, i.e. $[f] = f - f_1$, where f and f_1 are

† In this and in what follows the range of Latin indices is 1, 2, 3 and that of Greek indices is I, II. A repeated index implies summation unless stated otherwise.

values of f just behind and just ahead of the shock respectively. Then, the *shock conditions* derived by Kanwal (1960*b*) in the case of unsteady flows of conducting gases are

$$[\rho V_{nl}] = 0, \tag{1}$$

$$[H_{nl}] = 0, \tag{2}$$

$$\rho_1 V_{1nl}[u_i] = -\left[p + \frac{H^2}{8\pi}\right] N_i + \frac{H_{1nl}}{4\pi}[H_i], \tag{3}$$

$$[V_{nl} H_i] = H_{1nl}[u_i], \tag{4}$$

$$\rho_1 V_{1nl}\left[\frac{V^2}{2} + h + \frac{H^2}{8\pi\rho}\right] = \frac{H_{1nl}}{4\pi}[H_i u_i], \tag{5}$$

where $$V_i = u_i - GN_i, \quad V_{nl} \overset{\text{def}}{=} V_i N_i, \quad H_{nl} \overset{\text{def}}{=} H_i N_i, \quad h = \epsilon + p/\rho,$$

and all other symbols have their usual meanings.

Let us define the density-strength δ of the shock by

$$\delta = [\rho]/\rho_1. \tag{6}$$

In consequence of (6), (1), (3) and (4) we have

$$V_{nl} = \frac{V_{1nl}}{1+\delta}, \quad H_\beta = \frac{(1+\delta)(1-A_1)H_{1\beta}}{1-A_1-\delta A_1}, \tag{7}$$

where $$H_\beta \overset{\text{def}}{=} H_i x_{i\beta}, \quad x_{i\beta} \overset{\text{def}}{=} \partial x_i/\partial y^\beta, \quad A_1 = \frac{H_{1nl}^2}{4\pi\rho_1 V_{1nl}^2}.$$

2. Normal vorticity and current density

For simplicity, we assume that the flow upstream from the shock is uniform and known and the lines of curvature are the Gaussian co-ordinate curves on the shock surface.

By virtue of Weingarten's formula (Eisenhart 1947) for the derivative of N_i, we get

$$V_{1nl,\alpha} = -(K_\alpha u_{1\alpha} + G_{,\alpha}), \quad H_{1nl,\alpha} = -K_\alpha H_{1\alpha} \quad (\alpha \text{ unsummed}), \tag{8}$$

where K_α are normal curvatures of the shock surface and a comma followed by an index denotes partial differentiation with respect to the corresponding co-ordinate.

Normal vorticity and current density just behind the shock surface are given by

$$\omega_{nl} = \epsilon^{\alpha\beta} u_{\beta\ \alpha} \tag{9}$$

and

$$J_{nl} = \frac{1}{4\pi}\epsilon^{\alpha\beta} H_{\beta,\alpha} \tag{10}$$

respectively, where $\epsilon^{\alpha\beta}$ are the components of permutation tensor of the shock surface.

Multiplying (3) and (4) by $x_{i\beta}$, we get

$$\rho_1 V_{1nl}[u_\beta] = \frac{H_{1nl}}{4\pi}[H_\beta] \tag{11}$$

and

$$[V_{nl} H_\beta] = H_{1nl}[u_\beta]. \tag{12}$$

Differentiating (11) and (12) partially with respect to y^α and making use of (7)) (8), (9) and (10), we get

$$\rho V_{n/}\,\omega_{n/}\dagger - H_{n/}J_{n/} = \epsilon^{\alpha\beta}Y_{\alpha\beta}, \tag{13}$$

and

$$4\pi V_{n/}J_{n/} - H_{n/}\omega_{n/} = \epsilon^{\alpha\beta}Z_{\alpha\beta}, \tag{14}$$

where

$$Y_{\alpha\beta} = \frac{\delta\rho_1 A_1 V_{1n/}H_{1\beta}}{H_{1n/}(1-A_1-\delta A_1)}(K_\alpha u_{1\alpha}+G_{,\alpha}) - \frac{\delta H_{1\beta}K_\alpha H_{1\alpha}}{4\pi(1-A_1-\delta A_1)},$$

$$Z_{\alpha\beta} = \frac{\delta A_1 H_{1\beta}}{1-A_1-\delta A_1}(K_\alpha u_{1\alpha}+G_{,\alpha}) - \frac{\delta A_1 V_{1n/}K_\alpha H_{1\alpha}H_{1\beta}}{H_{1n/}(1-A_1-\delta A_1)} + \frac{(1-A_1)V_{1n/}H_{1\beta}\delta_{,\alpha}}{(1+\delta)(1-A_1-\delta A_1)}.$$

The equations (13) and (14), derived purely from the dynamical relations (3) and (4), provide two linear relations between normal vorticity and current density generated behind a magnetogasdynamic shock wave. The shock condition (5) due to the law of conservation of energy across the shock is not at all needed in the process and as such the derivation of normal vorticity and current density is independent of the form of the equation of state. Thus we conclude the following theorem.

Theorem 1. *Normal vorticity and current density generated behind an oblique hydromagnetic shock wave in 3-dimensional unsteady flows do not depend upon the thermodynamical behaviour of the fluid and can be derived from purely dynamical considerations.*

When an electrically conducting fluid flows in a magnetic field, electric currents are induced in it. These currents modify the field and the field itself exerts forces which modify the flow. If the magnetic permeability μ is approximately regarded as equal to unity and the displacement currents are ignored, each element of the fluid will experience a force **L**, called Lorentz force, which is proportional to the cross product of the current density vector **J** and the magnetic field vector **H**, i.e.

$$\mathbf{L} = \mathbf{J}\times\mathbf{H}. \tag{15}$$

The components of the current density vector are given by

$$J_i = \frac{1}{4\pi}\epsilon_{ijk}H_{k,j}. \tag{16}$$

If, behind the shock surface, the current density vector coincides with the magnetic field vector, we have

$$J_i = \lambda H_i, \tag{17}$$

where λ is proportionality factor.

Multiplying (17) by N_i we obtain

$$\lambda = J_{n/}/H_{n/}. \tag{18}$$

Solving (13) and (14) for $J_{n/}$ and substituting in (18), we get

$$\lambda = \frac{A_1\delta(1+\delta)}{2\pi H_{1n/}(1-A_1-\delta A_1)^2}\left(\frac{K_\alpha u_{1\alpha}+G_{,\alpha}}{V_{1n/}} - \frac{K_\alpha H_{1\alpha}}{H_{1n/}}\right)H_{1\beta}\epsilon^{\alpha\beta} + \frac{(1-A_1)H_{1\beta}\delta_{,\alpha}\epsilon^{\alpha\beta}}{4\pi H_{1n/}(1-A_1-\delta A_1)^2}. \tag{19}$$

Differentiating (17) with respect to x_i and using the facts div $\mathbf{J} = 0$ and div $\mathbf{H} = 0$, we get

$$H_i\lambda_{,i} = 0,$$

† Indices behind solidus do not obey summation convention.

which shows that the proportionality factor λ remains constant along the magnetic lines and the Lorentz force is zero. Thus we conclude the following theorem.

Theorem 2. *If behind the shock surface the current vector coincides with the magnetic field vector, the proportionality factor λ given by* (19) *remains constant along magnetic lines and the flow experiences no Lorentz force; conversely, if the flow-field behind the shock surface is Lorentz force free, the current lines coincide with magnetic lines and their proportionality factor λ is given by* (19).

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REFERENCES

EISENHART, L. P. 1947 *Introduction to Differential Geometry*. New York: Princeton University Press.

HAYES, W. D. 1957 *J. Fluid Mech.* **2**, 595–600.

KANWAL, R. P. 1960*a* *Proc. Roy. Soc.* A **257**, 263–8.

KANWAL, R. P. 1960*b* *J. Math. Mech.* **9**, 681–95.

LIGHTHILL, M. J. 1957 *J. Fluid Mech.* **2**, 1–32.

RAM, RISHI 1967 *Annales de la Société Scientifique de Bruxelles* **81**, no. I, 87–96.

RAM, RISHI & MISHRA, R. S. 1966 *Tensor* (*N.S.*) **17**, 279–87.

TRUESDELL, C. A. 1952 *J. Aero. Sci.* **19**, 826–8.